On the number of two-dimensional threshold functions

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A two-dimensional threshold function of k-valued logic can be viewed as coloring of the points of a $k \times k$ square lattice into two colors such that there exists a straight line separating points of different colors. For the number of such functions only asymptotic bounds are known. We give an exact formula for the number of two-dimensional threshold functions and derive more accurate asymptotics.

1 Introduction

A function f of n variables of k-valued logic is called a *threshold function* if it takes two value 0 and 1 and there exists a hyperplane separating the pre-images $f^{-1}(0)$ and $f^{-1}(1)$. Threshold functions have been studied from the perspectives of electrical engineering [6], neural networks [3], combinatorial geometry [1, 8, 4], and learning theory [11, 15].

Computing the number $P(k, n)$ of n-dimensional k-valued threshold functions turns out to be a hard problem, even in the case of $k = 2$. The number $P(2, n)$ corresponding to n-dimensional boolean threshold functions was studied in a number of publications [14, 7, 8, 18]. Despite of many efforts, the exact values of $P(2, n)$ is known only for $n \leq 8$ (sequence *A000609* in [12]). Asymptotic of $P(2, n)$ was found in [16, 17]. Computing $P(k, n)$ for $k > 2$ appears to be even a harder problem. Known results on the number $P(k, n)$ for $k > 2$ are mostly of asymptotic nature [5, 4]. A particular case of two-dimensional threshold functions (i.e., for $n = 2$) was studied in [11]. Shevchenko [10] states the following asymptotic bounds

$$\frac{3}{8\pi^2}k^4 \lesssim P(k, 2) \lesssim \frac{6}{\pi^2}k^4.$$

In this paper we prove an exact formula for $P(k, 2)$ and derive a more accurate asymptotic.

The paper is organized as follows. In Section 2 we give a rigorous definition and examples of two-dimesional threshold functions. The exact formula for $P(k, 2)$ is obtained in Section 3. Asymptotics for $P(k, 2)$ are given in Section 4. Finally, in Section 5 we discuss connection to teaching sets and pose related open problems.

2 Two-dimensional threshold functions

We consider the problem of computing of $P(k, 2)$ in a slightly more general form, allowing the arguments of two-dimensional threshold functions take different number of integer values. The precise definition follows.

Let $m,\ n \in \mathbb{N}$ be positive integers and $\mathcal{K}$ be a rectangular area on a plane bounded by the lines $x = 0,\ x = m,\ y = 0,\ y = n$ with an integer interior $\mathcal{K}_0$, i.e., $\mathcal{K} \stackrel{\text{def}}{=} [0, m] \times [0, n]$ and $\mathcal{K}_0 \stackrel{\text{def}}{=} \mathcal{K} \cap \mathbb{Z}^2$.

Definition 1. *A (two-dimensional)* threshold function *on $\mathcal{K}_0$ is a function $f : \mathcal{K}_0 \to \{0, 1\}$ not identically* 0 *or* 1 *such that there exists a line $\ell = ax + by + c \equiv 0$ satisfying*

$$f(x, y) = 0 \qquad \Longleftrightarrow \qquad ax + by + c \leq 0.$$

We say that the line ℓ defines *the threshold function f.*

Examples of threshold functions are given in Fig. 1.

Let $N(m, n)$ be the number of all threshold functions on $\mathcal{K}_0$. Our goal is to find an exact formula and an asymptotic for $N(m, n)$. That will immediately imply similar results for $P(k, 2)$ since $P(k, 2) = N(k - 1, k - 1)$.

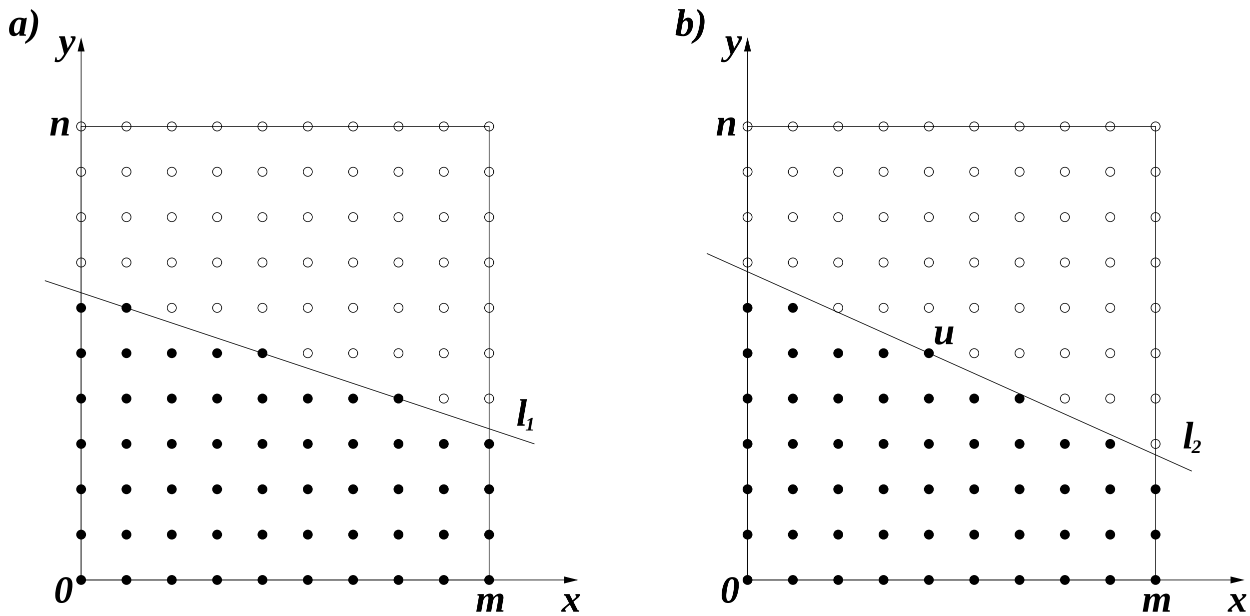

Figure 1: Examples of threshold functions defined by lines ℓ_1 and ℓ_2 (the filled dots correspond to zero values). a) The line ℓ_1 defines a stable threshold function. b) The line ℓ_2 defines an unstable threshold function with a vertex at u.

3 Exact formulas for $N(m,n)$ and $P(k,2)$

In this section we will prove the following theorem:

Theorem 2. *The total number of 2-D threshold functions is*

$$N(m,n) = 2\left(2mn + m + n + 2V(m,n)\right),$$

and, in particular,

$$P(k,2) = N(k-1,k-1) = 4\left(k^2 - k + V(k-1,k-1)\right),$$

where

$$V(m,n) \stackrel{\text{def}}{=} \sum_{\substack{i=1 \\ \gcd(i,j)=1}}^{\lceil m \rceil} \sum_{j=1}^{\lceil n \rceil} (m+1-i)(n+1-j).$$

3.1 Preliminary results

All threshold functions fall into two classes, depending on the value of $f(0,0)$. Between these two classes there is a natural bijection $f \mapsto 1 - f$. Hence, it is enough to consider a class $\mathfrak{M}$ of threshold functions with $f(0,0) = 0$. The total number of threshold functions is twice as many as $|\mathfrak{M}|$.

With each line $\ell = ax + by + c \equiv 0$ we associate a set of zeros of a threshold function defined by ℓ:
$$M(\ell) = M(a, b, c) = \{(x, y) \in \mathcal{K}_0 \mid ax + by + c \leq 0\}.$$
If the line ℓ defines a function from $\mathfrak{M}$, then $c \leq 0$. For the rest assume that this inequality always holds.

Note that if the line ℓ does not contain the origin (a *regular* line), then it uniquely defines a function from $\mathfrak{M}$. If the line ℓ contains the origin (a *singular* line), then in general it defines two functions from $\mathfrak{M}$. If a singular line ℓ' is the result of continuous motion of a regular line ℓ, then we will assume that ℓ' defines a function equal the limit of the functions defined by ℓ.

Definition 3. *Call a line $\ell = ax+by+c \equiv 0$* horizontal, *if $a = 0$;* vertical, *if $b = 0$; and* inclined, *if $a \neq 0$ and $b \neq 0$. An inclined line is called* positive *or* negative *depending on the sign of its slope $\frac{-a}{b}$.*

Definition 4. *Lines ℓ_1 and ℓ_2 are* equivalent *($\ell_1 \sim \ell_2$), if $M(\ell_1) = M(\ell_2)$. In other words, two lines are equivalent if they define the same threshold function.*

We prove (non-)equivalence of the lines ℓ_1 and ℓ_2 using (non-)emptiness of the symmetric difference $M(\ell_1) \triangle M(\ell_2)$, or both set differences $M(\ell_1) \setminus M(\ell_2)$ and $M(\ell_2) \setminus M(\ell_1)$.

Lemma 5. *For any line ℓ defining a threshold function from $\mathfrak{M}$, there exists an equivalent line ℓ' passing through at least one point from $\mathcal{K}_0$.*

Proof. Let the line $\ell = ax + by + c \equiv 0$ define some threshold function from $\mathfrak{M}$. If ℓ does not contain points from $\mathcal{K}_0$, then increasing c (i.e., translating ℓ towards the origin) we will find a line $\ell' = ax + by + c' \equiv 0$ such that ℓ' passes through at least one point from $\mathcal{K}_0$, and there are no points from $\mathcal{K}_0$ between ℓ' and ℓ. Then $\ell \sim \ell'$. □

Denote by $\mathfrak{L}$ the set of all lines that define functions from $\mathfrak{M}$ and pass through at least one point from $\mathcal{K}_0$. Lemma 5 implies that every function from $\mathfrak{M}$ is defined by some line from $\mathfrak{L}$.

Lemma 6. *Let $\ell_1,\ \ell_2 \in \mathfrak{L}$. If $\ell_1 \sim \ell_2$ and $\ell_1 \neq \ell_2$, then there exists a point $q \in \ell_1 \cap \ell_2 \cap \mathcal{K}$.*

Proof. Assume that the given lines do not have a common point within the rectangle $\mathcal{K}$. Since $\ell_1,\ \ell_2 \in \mathfrak{L}$, there exist points $u \in \ell_1 \cap \mathcal{K}_0$ and $v \in \ell_2 \cap \mathcal{K}_0$. Then either $u \in M(\ell_1) \setminus M(\ell_2)$ or $v \in M(\ell_2) \setminus M(\ell_1)$, a contradiction to $\ell_1 \sim \ell_2$. □

Definition 7. *A line $\ell \in \mathfrak{L}$ is called* stable *if it passes through at least two points from $\mathcal{K}_0$.*

An example of a stable line is given in Fig. 1a.

Lemma 8. *Let $\ell_1 \sim \ell_2$ be two equivalent lines passing through points $u \in \mathcal{K}_0$ and $v \in \mathcal{K}_0$ respectively. If $u \neq v$ then there exists a stable line ℓ_0 such that $u \in \ell_0$ and $\ell_1 \sim \ell_0 \sim \ell_2$.*

Proof. If ℓ_1 is stable, then the statement is trivial for $\ell_0 = \ell_1$. Assume that ℓ_1 is not stable.

Lemma 6 implies $\ell_1 \cap \ell_2 = q \in \mathcal{K}$. It is easy to see that $q \neq v$. On the other hand, if $q = u$ then $\ell_0 = \ell_2$ proves the lemma. Hence, assume $q \neq u$.

Consider a family of lines $\ell(t)$ passing through the points u and $q + (v - q)t$ for $t \in [0, 1]$. Note that $\ell(0) = \ell_1$. Let $t_0 > 0$ be a minimal value of t such that line $\ell(t)$ passes through a point from $\mathcal{K}_0$ different from u. Define $\ell_0 = \ell(t_0)$.

We will prove that the set $M(\ell(t))$ does not change as the parameter t changes from 0 till t_0. By the construction, $M(\ell') = M(\ell_1)$ for any intermediate line $\ell' = \ell(t)$, $t \in (0, t_0)$ and all points from $\ell_1 \cap \mathcal{K}_0$ different from u lie on the same side of ℓ' as the point $v \in M(\ell_1) = M(\ell_2)$. Hence, $M(\ell') = M(\ell_1)$. On the other hand, for any point $w \in \ell_0 \cap \mathcal{K}_0$, it is true that $w \in M(\ell_1) = M(\ell_2)$. Indeed, if $w \notin M(\ell_1) = M(\ell_2)$ then point w must lie on a ray of line ℓ_0 with origin u, which crosses line ℓ_2. But then points w and v lie at the same side of line ℓ_1, and, hence, $w \in M(\ell_1)$ contradicting to the assumption $w \notin M(\ell_1) = M(\ell_2)$.

Therefore, $M(\ell_0) = M(\ell(t_0)) = M(\ell(0)) = M(\ell_1)$, i.e, $\ell_0 \sim \ell_1$. □

Lemma 9. *If two equivalent lines ℓ_1 and ℓ_2 have a common point $u \in \mathcal{K}_0$, then $\ell_1 = \ell_2$.*

Proof. Suppose that a line ℓ_1 passes through points $u \neq v \in \mathcal{K}_0$, a line ℓ_2 passes through points $u \neq w \in \mathcal{K}_0$, and $\ell_1 \sim \ell_2$.

If the line ℓ_1 is vertical, then $u_x = v_x = c$. In this case the line ℓ_2 cannot be horizontal, otherwise the corresponding threshold function must be a null. From $w \in M(\ell_2) = M(\ell_1)$ it follows that $w_x \leq c$. If $w_x < c$, then $(c, w_y) \in M(\ell_1) \setminus M(\ell_2)$, a contradiction to $\ell_1 \sim \ell_2$. Therefore, $w_x = c$ and $\ell_2 = \ell_1$.

The other cases with a horizontal or vertical line are considered similarly. Assume that both lines ℓ_1, ℓ_2 are inclined.

If the point u lies on a border of rectangle $\mathcal{K}$, then either $v \notin M(\ell_2)$, or $w \notin M(\ell_1)$, a contradiction to $\ell_1 \sim \ell_2$. Hence, u is an internal point of rectangle $\mathcal{K}$.

It is easy to see that if the lines ℓ_1 and ℓ_2 have opposite signs, then the lines $x = u_x$ or $y = u_y$ contain a point from $M(\ell_1) \triangle M(\ell_2)$.

The remaining case to consider is the lines ℓ_1 and ℓ_2 having the same sign. Without loss of generality, assume that they are positive. Also assume that both lines cross the bottom side of rectangle $\mathcal{K}$, the other cases (e.g., crossing the left side or passing through $\mathbf{0}$) are considered similarly. Note that the case, when one line crosses the left side while the other crosses the bottom side, is impossible since it would imply $(0, n) \in M(\ell_1) \triangle M(\ell_2)$ contradicting to the equivalence of ℓ_1 and ℓ_2.

Suppose that the slopes of the lines are not equal. Without loss of generality, assume that the slope of ℓ_1 is less. Then $v_x < u_x < w_x$ and $v_y < u_y < w_y$. Hence, line ℓ passing through the points v and w is positive (Fig. 2a).

Let $z = \frac{v+w}{2}$ be a middle point of the interval $[v, w]$. Consider a point v' symmetric to v with respect to the point u. Then $v' \notin \mathcal{K}_0$, since otherwise $v' \in M(\ell_1) \setminus M(\ell_2)$ that is impossible due to $\ell_1 \sim \ell_2$. Hence, in particular, $v'_x = 2u_x - v_x > w_x$ that is equivalent to $u_x > z_x$. Similarly, for a point w' symmetric to the point w with respect to the point u, we have $u_y < z_y$. Since u is an integer point and z is a middle point of the interval with integer ends, a stronger inequality $z_y - u_y \geq \frac{1}{2}$ holds.

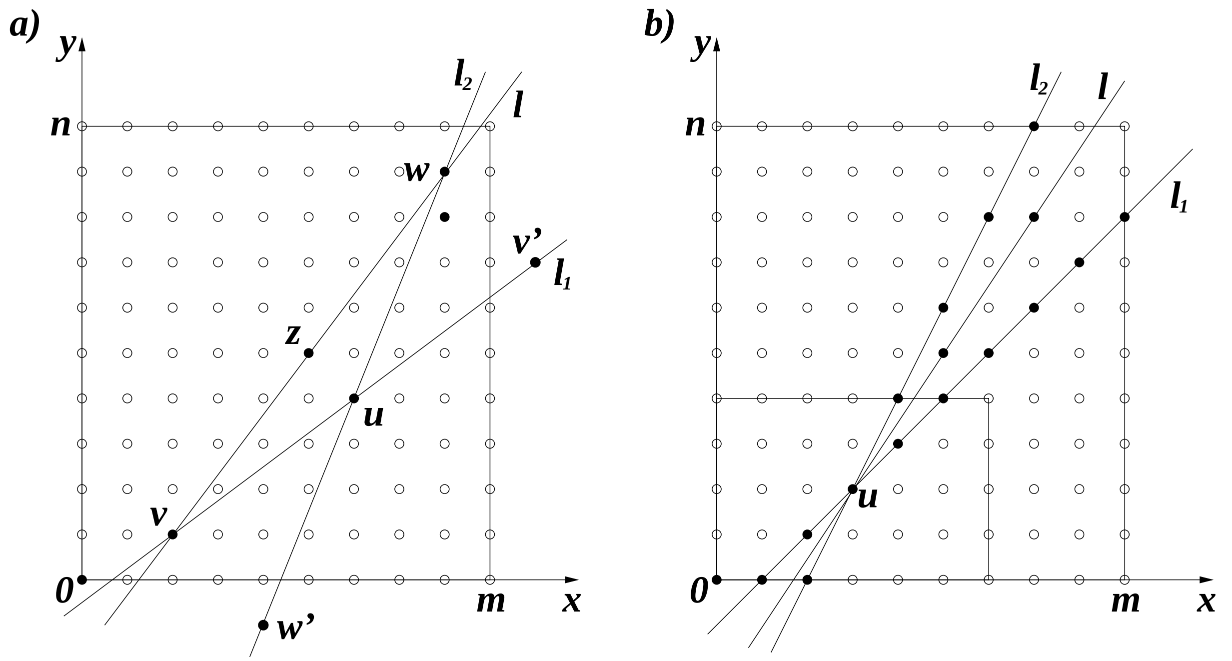

Figure 2: a) Point $(w_x, w_y - 1)$ belongs to $M(\ell_1) \setminus M(\ell_2)$ contradicting $\ell_1 \sim \ell_2$. b) Line ℓ lies between adjacent u-stable lines ℓ_1 and ℓ_2.

Consider an equation of the line ℓ_1 in the form $y_1(x) = \frac{u_y - v_y}{u_x - v_x}(x - v_x) + v_y$; and an equation of the line ℓ in the form $y(x) = \frac{w_y - v_y}{w_x - v_x}(x - v_x) + v_y$. Define a function

$$f(x) \stackrel{\text{def}}{=} y(x + v_x) - y_1(x + v_x) = \left(\frac{w_y - v_y}{w_x - v_x} - \frac{u_y - v_y}{u_x - v_x}\right)x.$$

Trivially, $f(kx) = kf(x)$ for all k.

Since the line ℓ_1 is positive, and $z_x < u_x$, then $y_1(z_x) < y_1(u_x) = u_y$. Hence, $f\left(\frac{w_x - v_x}{2}\right) = z_y - y_1(z_x) > z_y - u_y \geq \frac{1}{2}$. Linearity of $f(x)$ implies

$$w_y - y_1(w_x) = f(w_x - v_x) = 2f\left(\frac{w_x - v_x}{2}\right) > 1.$$

that is equivalent to $y_1(w_x) < w_y - 1$. Hence, $(w_x, w_y - 1) \in M(\ell_1) \setminus M(\ell_2)$ contradicting the equivalence of ℓ_1 and ℓ_2.

This contradiction proves that the slopes of ℓ_1 and ℓ_2 are equal. Since ℓ_1 and ℓ_2 have the common point u, Lemma 9 implies $\ell_1 = \ell_2$. □

Lemma 10. *If lines ℓ_1 and ℓ_2 are stable and $\ell_1 \sim \ell_2$, then $\ell_1 = \ell_2$.*

Proof. Assume that lines ℓ_1 and ℓ_2 are stable and $\ell_1 \sim \ell_2$, but $\ell_1 \neq \ell_2$. By Lemma 6, $\ell_1 \cap \ell_2 = q \in \mathcal{K}$.

If $q \in \mathcal{K}_0$, then Lemma 9 implies $\ell_1 = \ell_2$, a contradiction.

Assume $q \notin \mathcal{K}_0$. Let points $u, v \in \mathcal{K}_0$ be the closest to q lying on the lines ℓ_1 and ℓ_2 respectively. By Lemma 8 there exists a stable line ℓ_0 equivalent ℓ_1 such that $u \in \ell_0$. Lemma 9 applied to the lines ℓ_0 and ℓ_1 implies $\ell_0 = \ell_1$ that is impossible since ℓ_0 differs from ℓ_1 by the construction. This contradiction completes the proof. $\square$

Lemma 11. *Let the line ℓ_1 pass through a point $u \in \mathcal{K}_0$. If line ℓ_2 is stable and $\ell_1 \sim \ell_2$, then ℓ_2 passes through u as well.*

Proof. If line ℓ_1 is stable, then the statement immediately follows from Lemma 9. Hence, suppose that line ℓ_1 is not stable, i.e., $\ell_1 \cap \mathcal{K}_0 = \{u\}$.

According to Lemma 6, $\ell_1 \cap \ell_2 = q \in \mathcal{K}$. Let point $v \in \mathcal{K}_0$ be the closest to q on the line ℓ_2.

If $q \in \mathcal{K}_0$, i.e., $v = u = q$, then the lemma is proved.

Assume $q \notin \mathcal{K}_0$. By Lemma 8 there exists a stable line ℓ_0 such that $u \in \ell_0$ and $\ell_0 \sim \ell_1$. Applying Lemma 10 to the lines ℓ_0 and ℓ_2, we conclude that $\ell_0 = \ell_2$ and, hence, $u \in \ell_2$. $\square$

Definition 12. *A threshold function $f \in \mathfrak{M}$ is called* stable*, if there exists a stable line that defines f, and* unstable *otherwise.*

Examples of stable and unstable functions are given in Fig. 1.

3.2 Number of unstable functions

Definition 13. *For an unstable function f, define* vertex *as a point from $\mathcal{K}_0$ that lies on every line from $\mathfrak{L}$ defining f. Such point exists according to Lemma 11.*

Definition 14. *Let $u \in \mathcal{K}_0$. Denote by $\mathcal{K}_u$ a maximal subarea of $\mathcal{K}$ (rectangular in general) that is centrally symmetric with respect to the point u; and let $\overline{\mathcal{K}_u} = \mathcal{K} \setminus \mathcal{K}_u$. Denote by L_u the set of all lines passing through u; and denote by S_u a set of stable lines from L_u. Line $\ell \in S_u$ is called* u-stable *if ℓ passes through a point from $\mathcal{K}_u \cap \mathcal{K}_0$ different from u.*

Definition 15. *Let $u \in \mathcal{K}_0$. Lines $\ell_1,\ \ell_2 \in S_u$ are called* adjacent *if a line ℓ during the shortest rotation from a position ℓ_1 to a position ℓ_2 about the point u does not meet any other lines from S_u. Any line corresponding to an intermediate position of ℓ is said* lying between *the lines ℓ_1 and ℓ_2 (Fig. 2b).*

Lemma 16. *If line ℓ passing through $u \in \mathcal{K}_0$ lies between adjacent lines $\ell_1,\ \ell_2 \in S_u$, and $\ell \sim \ell'$ for some stable line ℓ', then either $\ell' = \ell_1$, or $\ell' = \ell_2$.*

Proof. Assume that ℓ' differs from ℓ_1 and ℓ_2. Let v_1 and v_2 be points from $\mathcal{K}_0$ different from u lying on the lines ℓ_1 and ℓ_2 respectively. Note that Lemma 11 implies $u \in \ell'$. During the shortest rotation (about the point u) to line ℓ', line ℓ necessary meets ℓ_t ($t = 1$ or 2). But then point v_t belongs to $M(\ell) \vartriangle M(\ell')$, and, hence, ℓ and ℓ' cannot be equivalent. This contradiction completes the proof. $\square$

For a point $u = \mathbf{0}$, there exists a unique unstable threshold function with a vertex at u. Namely, this function is equal to 0 only at the point $\mathbf{0}$, and is defined by any negative line passing through the point $\mathbf{0}$.

If $u \neq \mathbf{0}$ is a vertex of the rectangle $\mathcal{K}$, then there is no unstable threshold function with a vertex at u. It is easy to see that any line passing through u can be rotated about u into an equivalent stable line.

Exclude vertices of the rectangle $\mathcal{K}$ from $\mathcal{K}_0$, and denote the rest of the points by $\mathcal{K}'_0$.

Lemma 17. *Let $u \in \mathcal{K}'_0$, and ℓ_0 be a line passing through points $\mathbf{0}$ and u. Then the number of unstable threshold functions with a vertex at u is equal to the number of u-stable lines, if the line ℓ_0 is u-stable; and one less otherwise.*

Proof. Suppose that a line $\ell \in L_u \setminus S_u$ lies between adjacent lines $\ell_1,\ \ell_2 \in S_u$. Note that ℓ cannot be equivalent to any u-stable line, since the latter contains a pair of symmetric (with respect to u) points, one of which does not belong to $M(\ell)$. Hence, if both lines ℓ_1 and ℓ_2 are u-stable, then ℓ is not equivalent to either of them, and by Lemma 16 it must define an unstable threshold function.

Therefore, the statement is true when $\mathcal{K}_u = \mathcal{K}$ (e.g., u is a center of the rectangle $\mathcal{K}$). We exclude this case for the rest of the proof.

We split the set L_u into two:

$$L'_u \stackrel{\text{def}}{=} \{\ell \in L_u \mid \ell \cap \overline{\mathcal{K}_u} = \emptyset\};$$
$$L''_u \stackrel{\text{def}}{=} \{\ell \in L_u \mid \ell \cap \overline{\mathcal{K}_u} \neq \emptyset\}.$$

Unstable lines from L'_u have to lie between two adjacent stable lines from L'_u (obviously, they are u-stable as well). Hence, the number of distinct unstable threshold functions defined by lines from L'_u equals the number of u-stable lines in L'_u minus 1.

For any unstable line $\ell \in L''_u$, call *free end* a ray starting at u which crosses area $\overline{\mathcal{K}_u}$. Consider a rotation of ℓ about u such that free end of ℓ moves towards the point $\mathbf{0}$ following the shortest arc. Without loss of generality we assume that during such rotation ℓ first meets a stable line ℓ_1. Then ℓ can be equivalent only to ℓ_1 since the line ℓ_2 is either u-stable, or passes through a point from $\overline{\mathcal{K}_u} \cap \mathcal{K}_0$ not belonging to $M(\ell)$. But such equivalence takes place if and only if ℓ_1 is not u-stable. Hence, counting of unstable threshold functions with a vertex at u corresponds to counting of u-stable lines. The number of unstable threshold functions defined by lines from L''_u equals the number of u-stable lines in L''_u.

Note that if the line ℓ_0 is u-stable, it will be counted two times. In this case the number of unstable threshold functions defined by lines from L''_u is greater by 1 as compared to the number of u-stable lines in L''_u.

Hence, the total number of unstable threshold functions with a vertex at u is equal to the number of u-stable lines when the line ℓ_0 is u-stable; or less by 1 otherwise. □

Definition 18. *Let*

$$U(p,q) \stackrel{\text{def}}{=} \#\{(a,b) \in \mathbb{Z}^2 \mid 1 \leq a \leq p,\ 1 \leq b \leq q,\ \gcd(a,b) = 1\}.$$

or, equivalently,

$$U(p,q) = \sum_{i=1}^{p} \sum_{\substack{j=1 \\ \gcd(i,j)=1}}^{q} 1. \tag{1}$$

Note that values $U(k,k)$ related to the probability of two random numbers from $[1,k]$ being co-prime are well studied (see also sequence *A018805* in [12]).

Lemma 19. *The number of points $u \in \mathcal{K}_0'$ such that the line ℓ_0 is not u-stable equals $U(m,n)-1$.*

Proof. If the line ℓ_0 passes through exactly $k+1$ points $\mathbf{0} = u_0, u_1, \ldots, u_k \in \mathcal{K}_0$ listed in the order of increasing distance from the point $\mathbf{0}$, then ℓ_0 is u_i-stable for $i = 1, \ldots, k-1$ and is not u_k-stable. Therefore, the number of points $u \in \mathcal{K}_0'$ for which the line ℓ_0 is not u-stable equals the number of stable lines passing through the point $\mathbf{0}$, excluding three lines passing through the vertices of $\mathcal{K}$. □

Definition 20. *For real numbers t and k define a function*

$$V(t,k) \stackrel{\text{def}}{=} \sum_{\substack{i=1 \\ \gcd(i,j)=1}}^{\lceil t \rceil} \sum_{j=1}^{\lceil k \rceil} (t+1-i)(k+1-j). \tag{2}$$

Lemma 21. *For all $t \in \mathbb{R}, k \in \mathbb{N}$ the following holds*

i) $V(t,k-1) + V(t,k) = 2V\left(t, k - \frac{1}{2}\right)$.

For all $t, k \in \mathbb{N}$ the following holds

ii) $V(t,k) = \sum\limits_{p=1}^{t} \sum\limits_{q=1}^{k} U(p,q)$;

iii) $\sum\limits_{p=1}^{t} U(p,k) = V(t,k) - V(t,k-1)$;

iv) $U(t,k) = V(t,k) - V(t,k-1) - V(t-1,k) + V(t-1,k-1)$;

v) $U(t,k) + 2\left(V(t,k-1) + V(t-1,k)\right) = 4V\left(t - \frac{1}{2}, k - \frac{1}{2}\right)$.

Proof. Let $t \in \mathbb{R}, k \in \mathbb{N}$. To prove i) we use formula (2)

$$V(t,k-1) + V(t,k) = \sum_{\substack{i=1 \\ \gcd(i,j)=1}}^{\lceil t \rceil} \sum_{j=1}^{k-1} (t+1-i)(k-j) +$$
$$\sum_{\substack{i=1 \\ \gcd(i,j)=1}}^{\lceil t \rceil} \sum_{j=1}^{k} (t+1-i)(k+1-j) = \sum_{\substack{i=1 \\ \gcd(i,j)=1}}^{\lceil t \rceil} \sum_{j=1}^{k} (t+1-i)(2k+1-2j) =$$
$$2 \sum_{\substack{i=1 \\ \gcd(i,j)=1}}^{\lceil t \rceil} \sum_{j=1}^{k} (t+1-i)\left(k + \frac{1}{2} - j\right) = 2V\left(t, k - \frac{1}{2}\right).$$

To prove ii) we use formulae (1) and (2)

$$\sum_{p=1}^{t}\sum_{q=1}^{k} U(p,q) = \sum_{p=1}^{t}\sum_{q=1}^{k}\sum_{\substack{i=1 \\ \gcd(i,j)=1}}^{p}\sum_{j=1}^{q} 1 = \sum_{\substack{i=1 \\ \gcd(i,j)=1}}^{t}\sum_{j=1}^{k}\sum_{p=i}^{t}\sum_{q=j}^{k} 1 =$$
$$\sum_{\substack{i=1 \\ \gcd(i,j)=1}}^{t}\sum_{j=1}^{k} (t+1-i)(k+1-j) = V(t,k).$$

To prove iii) we use formula ii)

$$V(t,k) - V(t,k-1) = \sum_{p=1}^{t}\sum_{q=1}^{k} U(p,q) - \sum_{p=1}^{t}\sum_{q=1}^{k-1} U(p,q) = \sum_{p=1}^{t} U(p,k).$$

Using iii), prove iv)

$$U(t,k) = \sum_{p=1}^{t} U(p,k) - \sum_{p=1}^{t-1} U(p,k) = V(t,k) - V(t,k-1) - V(t-1,k) + V(t-1,k-1).$$

Finally, to prove equality v) we use formulae iv) and i)

$$\begin{gathered} U(t,k) + 2\,(V(t,k-1) + V(t-1,k)) = \\ V(t,k) + V(t,k-1) + V(t-1,k) + V(t-1,k-1) = \\ = 2V\left(t, k-\tfrac{1}{2}\right) + 2V\left(t-1, k-\tfrac{1}{2}\right) = 4V\left(t-\tfrac{1}{2}, k-\tfrac{1}{2}\right). \end{gathered}$$

□

Theorem 22. *The number of unstable threshold functions in $\mathfrak{M}$ is*

$$2mn - U(m,n) + 8V\left(\frac{m-1}{2}, \frac{n-1}{2}\right).$$

Proof. Note that if $u_x \le \frac{m}{2}$ and $u_y \le \frac{n}{2}$, then the number of u-stable inclined lines equals $2U(u_x, u_y)$. If u lie on a side of $\mathcal{K}$, then the line passing through that side is the only u-stable line. If u is an inner point of $\mathcal{K}$, then both vertical and horizontal as well as inclined lines passing through u are u-stable.

Let us count the number of all u-stable lines for $u \in \mathcal{K}_0'$. Despite that the counting depends on the evenness of the integers m and n, we will show that the result is the same in all cases and equals

$$2(mn-1) + 8V\left(\frac{m-1}{2}, \frac{n-1}{2}\right). \tag{3}$$

Consider all cases of m and n evenness.

If both m and n are odd, then Lemmas 17 and 21.ii imply that the number of all u-stable lines for $u \in [0, \frac{m-1}{2}] \times [0, \frac{n-1}{2}] \setminus \{\mathbf{0}\}$, equals

$$\sum_{i=1}^{\frac{m-1}{2}} 1 + \sum_{j=1}^{\frac{n-1}{2}} 1 + \sum_{i=1}^{\frac{m-1}{2}}\sum_{j=1}^{\frac{n-1}{2}} (2U(i,j)+2) = \frac{mn-1}{2} + 2V\left(\frac{m-1}{2}, \frac{n-1}{2}\right).$$

Due to the symmetry, the total number of u-stable lines for $u \in \mathcal{K}_0'$ is four times as many.

Now let m be even and n be odd. Then the number of all u-stable lines for $u \in [0, \frac{m}{2} - 1] \times [0, \frac{n-1}{2}] \setminus \{\mathbf{0}\}$ equals

$$\sum_{i=1}^{\frac{m}{2}-1} 1 + \sum_{j=1}^{\frac{n-1}{2}} 1 + \sum_{i=1}^{\frac{m}{2}-1} \sum_{j=1}^{\frac{n-1}{2}} (2U(i,j) + 2) = \frac{mn - n - 1}{2} + 2V\left(\frac{m}{2} - 1, \frac{n-1}{2}\right).$$

Quadruplicated this number is the number of all u-stable lines for $u \in \mathcal{K}_0'$, except those lying on a line $x = \frac{m}{2}$. The number of u-stable lines for u on that line due to property 21.iii can be counted as

$$2\left(1 + \sum_{j=1}^{\frac{n-1}{2}} \left(2U(\frac{m}{2}, j) + 2\right)\right) = 2n + 4\left(V\left(\frac{m}{2}, \frac{n-1}{2}\right) - V\left(\frac{m}{2} - 1, \frac{n-1}{2}\right)\right).$$

Hence, the total number of u-stable lines for $u \in \mathcal{K}_0'$ equals

$$\begin{gathered} 4\left(\tfrac{mn-n-1}{2} + 2V\left(\tfrac{m}{2} - 1, \tfrac{n-1}{2}\right)\right) + 2n + 4\left(V\left(\tfrac{m}{2}, \tfrac{n-1}{2}\right) - V\left(\tfrac{m}{2} - 1, \tfrac{n-1}{2}\right)\right) = \\ = 2(mn-1) + 4\left(V\left(\tfrac{m}{2}, \tfrac{n-1}{2}\right) + V\left(\tfrac{m}{2} - 1, \tfrac{n-1}{2}\right)\right) = 2(mn-1) + 8V\left(\tfrac{m-1}{2}, \tfrac{n-1}{2}\right). \end{gathered}$$

Here we used the property 21.i.

A case of odd m and even n is considered similarly.

Finally, let both m and n be even. The number of all u-stable lines for $u \in [0, \frac{m}{2} - 1] \times [0, \frac{n}{2} - 1] \setminus \{\mathbf{0}\}$ equals

$$\sum_{i=1}^{\frac{m}{2}-1} 1 + \sum_{j=1}^{\frac{n}{2}-1} 1 + \sum_{i=1}^{\frac{m}{2}-1} \sum_{j=1}^{\frac{n}{2}-1} (2U(i,j) + 2) = \frac{mn - m - n}{2} + 2V\left(\frac{m}{2} - 1, \frac{n}{2} - 1\right).$$

Quadruplicated this number equals the number of all u-stable lines for $u \in \mathcal{K}_0'$, except for those lying on the lines $x = \frac{m}{2}$ and $y = \frac{n}{2}$. Because of the properties 21.iii and 21.v, the number of u-stable lines for u on these lines equals

$$\begin{gathered} 2\left(1 + 1 + \sum_{i=1}^{\frac{m}{2}-1} \left(2U(i, \tfrac{n}{2}) + 2\right) + \sum_{j=1}^{\frac{n}{2}-1} \left(2U(\tfrac{m}{2}, j) + 2\right)\right) + 2U\left(\tfrac{m}{2}, \tfrac{n}{2}\right) + 2 = \\ 2(m + n - 1) + 2U\left(\tfrac{m}{2}, \tfrac{n}{2}\right) + \\ 4\left(V\left(\tfrac{m}{2} - 1, \tfrac{n}{2}\right) + V\left(\tfrac{m}{2}, \tfrac{n}{2} - 1\right) - 2V\left(\tfrac{m}{2} - 1, \tfrac{n}{2} - 1\right)\right) = \\ 2(m + n - 1) + 2V\left(\tfrac{m-1}{2}, \tfrac{n-1}{2}\right) - 8V\left(\tfrac{m}{2} - 1, \tfrac{n}{2} - 1\right). \end{gathered}$$

Summing up the results, we get the already known expression (3).

Lemmas 17 and 19 imply that the total number of unstable threshold function with a vertex in $\mathcal{K}_0'$ is equal to (3) minus $U(m, n) - 1$. Finally, we need to add 1 for the single unstable threshold function with a vertex at $\mathbf{0}$. ◻

3.3 Number of stable threshold functions

Theorem 23. *The number of stable threshold functions in $\mathfrak{M}$ equals*

$$m+n+U(m,n)+2V(m,n)-8V\left(\frac{m-1}{2},\frac{n-1}{2}\right).$$

Proof. Consider any stable line ℓ of a positive slope passing through $\mathbf{0}$. Let point $(a,b) \in l \cap \mathcal{K}_0$ be the closest to $\mathbf{0}$ implying $\gcd(a,b)=1$.

Consider all stable lines parallel to ℓ. Every such line is defined by a pair of points $(x,y),(x+a,y+b) \in \mathcal{K}_0$ on it, where $(x,y) \in \mathcal{K}_0$ is the closest point to $\mathbf{0}$. Such pairs are uniquely defined by the following constraints

$$\begin{cases} x<a \text{ or } y<b, \\ x+a \le m, \\ y+b \le n. \end{cases}$$

Let P be the set of all points $(x,y) \in \mathcal{K}_0$ satisfying these constraints. Then the stable lines parallel to ℓ and the elements of P are in one-to-one correspondence.

If $a > \frac{m}{2}$ or $b > \frac{n}{2}$, then the set P equals $P_1 \stackrel{\text{def}}{=} \{(x,y) \in \mathcal{K}_0 \mid 0 \le x \le m-a,\ 0 \le y \le n-b\}$. In this case the number of stable lines parallel to ℓ is $(m+1-a)(n+1-b)$.

If $a \le \frac{m}{2}$ and $b \le \frac{n}{2}$, then the set P equals

$$P_1 \setminus \{(x,y) \in \mathcal{K}_0 \mid a \le x \le m-a,\ b \le y \le n-b\}.$$

Hence, in this case the number of stable lines parallel to ℓ is less then before by

$$(m+1-2a)(n+1-2b) = 4\left(\frac{m-1}{2}+1-a\right)\left(\frac{n-1}{2}+1-b\right).$$

Summing over all pairs (a,b) fulfilling the constraints and using formula (2), we get that the total number of stable lines of a positive (negative) slope is $V(m,n)-4V\left(\frac{m-1}{2},\frac{n-1}{2}\right)$. The total number of inclined stable lines is twice as many as this number.

Since each inclined stable line passing through $\mathbf{0}$ defines two distinct threshold functions, the result should be increased by the number of such lines, i.e., by $U(m,n)$.

Finally, taking into account m vertical lines $x=i,\ i=\overline{0,m-1}$ and n horizontal lines $y=j,\ j=\overline{0,n-1}$, we complete the proof. □

Summing the results of Theorems 22 and 23 gives the number of threshold functions in $\mathfrak{M}$. The total number of threshold functions $N(m,n)$ is as twice as many. Theorem 2 is proved.

4 Asymptotic of $N(m,n)$

Theorem 24. *For $m \ge n$, the following asymptotics holds*

$$N(m,n) = \frac{6}{\pi^2}m^2n^2 + O(m^2 n \ln n)$$

$$N(m,n) = 2((n+1)C_n - B_n)m^2 + O(mn^3)$$

where $\phi(t)$ is the totient function and

- $B_k \stackrel{\text{def}}{=} \sum\limits_{i=1}^{k} \phi(i) = \frac{3}{\pi^2}k^2 + O(k \ln k)$ *(Dirichlet's Theorem, see [9]);*
- $C_k \stackrel{\text{def}}{=} \sum\limits_{i=1}^{k} \frac{\phi(i)}{i} = \frac{6}{\pi^2}k + O(\ln k)$ *(see [9]).*

Lemma 25. *Let $s \geq 0$. Then*

$$\sum_{t=1}^{k} \frac{1}{t} = O(\ln k);$$

$$\sum_{t=k+1}^{\infty} \frac{1}{t^s} = O\left(\frac{1}{k^{s-1}}\right), \quad (s \neq 1);$$

$$\sum_{t=1}^{k} t^s \ln t = O\left(k^{s+1} \ln k\right);$$

$$\sum_{t=1}^{k} t^s = \frac{k^{s+1}}{s+1} + O(k^s) = O(k^{s+1}).$$

Proof. The statement follows from integral estimates of the sums. □

Theorem 26. *For $t \geq k$, the following inequality holds*

$$U(t,k) = U(k,t) = \frac{6}{\pi^2}tk + O(t \ln k);$$

Proof. Note that there exist exactly $\lfloor m/p \rfloor$ positive integers not exceeding m, which are divisible by p. Hence, there are $\lfloor t/p \rfloor \lfloor k/p \rfloor$ pairs (a,b), $1 \leq a \leq t$, $1 \leq b \leq k$, whose greatest common divider is divisible by p. The inclusion-exclusion principle [13] for the number of pairs (a,b) with $gcd(a,b)$ not divisible by any prime p (i.e., $gcd(a,b) = 1$) gives an exact formula

$$U(t,k) = \sum_{s=1}^{\infty} \mu(s) \left\lfloor \frac{t}{s} \right\rfloor \left\lfloor \frac{k}{s} \right\rfloor,$$

where $\mu(s)$ is Möbeus function.

Approximate $U(t,k)$ with a function

$$\hat{U}(t,k) \stackrel{\text{def}}{=} \sum_{s=1}^{\infty} \mu(s) \frac{t}{s}\frac{k}{s} = tk \sum_{s=1}^{\infty} \frac{\mu(s)}{s^2} = tk\frac{6}{\pi^2}.$$

and bound an absolute value of the difference $U(t,k) - \hat{U}(t,k)$ as follows

$$\left|U(t,k) - \hat{U}(t,k)\right| = \left|\sum_{s=1}^{\infty} \mu(s) \left(\left\lfloor \frac{t}{s} \right\rfloor \left\lfloor \frac{k}{s} \right\rfloor - \frac{t}{s}\frac{k}{s}\right)\right| \leq \sum_{s=1}^{\infty} d(t,k,s),$$

where

$$d(t,k,s) \stackrel{\text{def}}{=} \frac{tk}{s^2} - \left\lfloor \frac{t}{s} \right\rfloor \left\lfloor \frac{k}{s} \right\rfloor.$$

If $s > k$, then $d(t,k,s) = \frac{tk}{s^2}$; otherwise (i.e., $s \le k$)

$$d(t,k,s) < \frac{tk}{s^2} - \left(\frac{t}{s} - 1\right)\left(\frac{k}{s} - 1\right) = \frac{t+k}{s} - 1 < \frac{2t}{s}.$$

Applying Lemma 25, we finally have

$$\left|U(t,k) - \hat{U}(t,k)\right| < \sum_{s=1}^{\infty} d(t,k,s) < \sum_{s=1}^{k} \frac{2t}{s} + \sum_{s=k+1}^{\infty} \frac{tk}{s^2} = O(t \ln k).$$

□

Note that Dirichlet's Theorem is a particular case of Theorem 26 for $t = k$.

Theorem 27. *For $t \ge k$, the following asymptotic holds*

$$V(t,k) = V(k,t) = \frac{3}{2\pi^2} t^2 k^2 + O(t^2 k \ln k).$$

Proof. We use formula 21.ii and property $U(i,j) = U(j,i)$ as follows.

$$V(t,k) = \sum_{i=1}^{t} \sum_{j=1}^{k} U(i,j) = 2 \sum_{i=1}^{k} \sum_{j=1}^{i-1} U(i,j) + \sum_{i=1}^{r} U(i,i) + \sum_{i=k+1}^{t} \sum_{j=1}^{k} U(i,j).$$

According to Lemmas 25 and 26,

$$\sum_{i=1}^{k} \sum_{j=1}^{i-1} U(i,j) = \sum_{i=1}^{k} \sum_{j=1}^{i-1} \left(\frac{6}{\pi^2} ij + O(i \ln j)\right) = \frac{6}{\pi^2} \sum_{i=1}^{k} \left(i \sum_{j=1}^{i-1} j + O\left(i \sum_{j=1}^{i-1} \ln j\right)\right) =$$

$$= \frac{6}{\pi^2} \sum_{i=1}^{k} \left(i \frac{i^2}{2} + O(i^2 \ln i)\right) = \frac{3}{\pi^2} \sum_{i=1}^{k} i^3 + O\left(\sum_{i=1}^{k} i^2 \ln i\right) = \frac{3}{4\pi^2} k^4 + O(k^3 \ln k).$$

Similarly,

$$\sum_{i=1}^{k} U(i,i) = O\left(\sum_{i=1}^{k} i^2\right) = O(k^3);$$

$$\sum_{i=k+1}^{t} \sum_{j=1}^{k} U(i,j) = \sum_{i=k+1}^{t} \sum_{j=1}^{k} \left(\tfrac{6}{\pi^2} ij + O(i \ln j)\right) =$$
$$\sum_{i=k+1}^{t} \left(\tfrac{6}{\pi^2} i \tfrac{k^2}{2} + O(ik \ln k)\right) = \tfrac{3}{2\pi^2} k^2 (t^2 - k^2) + O(t^2 k \ln k).$$

Therefore,

$$V(m,k) = 2\left(\tfrac{3}{4\pi^2} k^4 + O(k^3 \ln k)\right) + O(k^3) + \tfrac{3}{2\pi^2} k^2 (m^2 - k^2) + O(m^2 k \ln k) =$$
$$\tfrac{3}{2\pi^2} m^2 k^2 + O(m^2 k \ln k).$$

□

Theorem 28. *For $t \geq k$, the following inequality holds*

$$U(t,k) = U(k,t) = C_k t + O(k^2).$$

Proof. We use formula (1)

$$U(k,t) = \sum_{\substack{i=1 \\ \gcd(i,j)=1}}^{k} \sum_{j=1}^{t} 1 = \sum_{i=1}^{k} \left(\sum_{s=0}^{\lfloor \frac{t}{i} \rfloor - 1} \sum_{\substack{j=si+1 \\ \gcd(j,i)=1}}^{si+i} 1 + \sum_{\substack{j=\lfloor t/i \rfloor i+1 \\ \gcd(j,i)=1}}^{t} 1 \right) =$$

$$\sum_{i=1}^{k} \left\lfloor \frac{t}{i} \right\rfloor \phi(i) + O(k^2) = \sum_{i=1}^{k} \frac{t}{i}\phi(i) + O(k^2).$$

□

Theorem 29. *For $t \geq k$, the following asymptotic holds*

$$V(t,k) = V(k,t) = \frac{(k+1)C_k - B_k}{2} t^2 + O(tk^3).$$

Proof. We use formula (2)

$$V(k,t) = \sum_{\substack{i=1 \\ \gcd(i,j)=1}}^{k} \sum_{j=1}^{t} (k+1-i)(t+1-j) =$$

$$\sum_{i=1}^{k} (k+1-i) \left(\sum_{s=0}^{\lfloor \frac{t}{i} \rfloor - 1} \sum_{\substack{j=1 \\ \gcd(j,i)=1}}^{i} (t+1-si-j) + \sum_{\substack{j=1 \\ \gcd(j,i)=1}}^{t \bmod i} (t \bmod i + 1 - j) \right).$$

Neglecting terms of order $O(tk^3)$, we have

$$V(k,t) = \sum_{i=1}^{k} (k+1-i) \sum_{s=0}^{\lfloor \frac{t}{i} \rfloor - 1} \sum_{\substack{j=1 \\ \gcd(j,i)=1}}^{i} (t - si) + O(tk^3) =$$

$$\frac{t^2}{2} \sum_{i=1}^{k} (k+1-i) \frac{\phi(i)}{i} + O(tk^3) = \frac{(k+1)C_k - B_k}{2} t^2 + O(tk^3).$$

□

Theorem 24 now follows from Theorems 2, 27, and 29.

5 Relation to teaching sets

A *teaching set* [2, 11, 15] of a threshold function $f : \mathcal{K}_0 \to \{0,1\}$ is a subset $T \subset \mathcal{K}_0$ such that for any other threshold function $g \neq f$ there exists $t \in T$ such that $g(t) \neq f(t)$. A teaching set of minimal cardinality is called *minimum teaching set*. A minimum teaching set of a two-dimensional threshold function consists of either 3 or 4 points [11].

It is easy to see that an unstable threshold function f has a teaching set of size 3. Indeed, let $u \in \mathcal{K}_0$ be a vertex of f and $\ell \ni u$ be a line defining f. Then ℓ lies between two

adjacent stable lines ℓ_1 and ℓ_2 passing respectively through some points $u_1 \neq u$ and $u_2 \neq u$ with $f(u_1) = f(u_2) \neq f(u)$. Then $\{u, u_1, u_2\}$ forms a teaching set of f.

However, the minimum size of a teaching set T of a stable threshold function f may be 3 or 4. Namely, $|T| = 3$ if the complement threshold function $\tilde{f}(x, y) = 1 - f(m - x, m - y)$ is unstable; and $|T| = 4$ if $\tilde{f}(x, y)$ is stable. Therefore, the stable threshold functions can be partitioned into two classes depending on the size of a minimal teaching set. Unfortunately, our results do not allow to compute the number of threshold functions in each class, that we pose as an open problem.

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